The Homotopy-Perturbation Method for Solving Klein-Gordon-Type Equations with Unbounded Right-Hand Side

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The approximate and/or exact solutions of the generalized Klein-Gordon- and sine-Gordon-type equations are obtained. We introduce a new type of initial conditions to extend the class of solvable problems.

Key words: Singularity; Initial Value Problems; Homotopy-Perturbation Method; Klein-Gordon Equation.

1. Introduction

We consider the generalization of Klein-Gordon and sine-Gordon equations, respectively,

$$u_{tt} - u_{xx} + b_1 u + b_2 g(u) = f(x,t) + h_{tt}(x,t) \quad (1)$$

and

$$u_{tt} - u_{xx} + g(u) = f(x,t) + h_{tt}(x,t), \quad (2)$$

where u is a function of x and t, g is a nonlinear function, and f and h are known differentiable functions. We focus on the unbounded case of $h_{tt}(x,t)$.

The Klein-Gordon and sine-Gordon equations model many problems in classical and quantum mechanics, solitons and condensed matter physics. Numerical solutions of Klein-Gordon equations and sine-Gordon equations have been investigated considerably in the last few years. Ablowitz and Herbst [1] presented the numerical results of the sine-Gordon equation. Ablowitz et al. [2] investigated the numerical behaviour of a double-discrete, completely integrable discretization of the sine-Gordon equation. Kaya [3] used the modified decomposition method to obtain approximate analytical solutions of the sine-Gordon equation. In [4], four finite difference schemes for approximating the nonlinear Klein-Gordon equation were discussed. Wazwaz [5] used the tanh method to obtain the exact solution of the sine-Gordon equation.

The purpose of the presented paper is to extend the class of solvable Klein-Gordon- and sine-Gordon-type equations by introducing the new type of initial conditions. We apply the homotopy-perturbation method (HPM), first proposed by He [6] and further developed and improved by He [7 – 11], to get the approximate solutions. He considered mainly the differential equations with analytical right-hand side (see for example [10], p. 1172).

2. The Extended Problem

In this section, we will introduce a new, extended form of initial conditions and apply the modified HPM [12] to get an approximate solution. We introduce the initial conditions as

$$\begin{aligned} \lim_{t\to 0} u(x,t) &= \lim_{t\to 0} [\varphi(x,t) + h(x,t)], \\ \lim_{t\to 0} u_t(x,t) &= \lim_{t\to 0} [\varphi_t(x,t) + h_t(x,t)], \end{aligned} \quad (3)$$

where $\varphi(x,t)$ and $h(x,t)$ are given functions and $\varphi(x,t)$ is bounded. The case $h(x,t) = 0$, $\lim_{t\to 0} \varphi(x,t) = \varphi(x,0)$, and $\lim_{t\to 0} \varphi_t(x,t) = \varphi_t(x,0)$ (that is, φ and φ_t are continuous functions) corresponds to the standard Klein-Gordon and sine-Gordon equations. The case $\varphi = h = 0$ corresponds to the standard [12] initial value problem (IVP) $u(x,0) = 0$, $u_t(x,0) = 0$.

Let us rewrite (1) as

$$Lu + Nu = f(x,t) + h_{tt}(x,t), \quad (4)$$

where L and N are, respectively, the linear and nonlinear operators.

According to the HPM, we construct a homotopy which satisfies the relation

$$\begin{aligned} H(u,p) = Lu - Lv_0 + pLv_0 \\ + p[Nu - f(x,t) - h_{tt}(x,t)] = 0, \end{aligned} \quad (5)$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions. When we put $p=0$ and $p=1$ in (5), we obtain

$$H(u,0)=Lu-Lv_0=0 \text{ and } \\ H(u,1)=Lu+Nu-f(x,t)-h_{tt}(x,t)=0, \quad (6)$$

which are the linear and nonlinear original equations, respectively.

We introduce an alternative way of choosing the initial approximations, that is

$$v_0=\varphi(x,t)+t\varphi_t(x,t)+\delta L^{-1}(f(x,t)+h_{tt}(x,t)), \quad (7)$$

where $\delta=1$, if $h_{tt}(x,t)$ is unbounded in t for fixed x, and $\delta=0$, if $h_{tt}(x,t)$ is bounded or unbounded only in x for fixed t. In the HPM, the solution of (4) is expressed as

$$u(x,t)=u_0(x,t)+pu_1(x,t)+p^2u_2(x,t)+\cdots. \quad (8)$$

Hence, the approximate solution of (4) can be expressed as a series of powers of p, i. e.

$$u=\lim_{p\to 1}u=u_0+u_1+u_2+\cdots. \quad (9)$$

3. Applications

In order to assess both the applicability and the accuracy of the procedure in case of unbounded h_{tt}, some test examples are considered.

3.1. Example 1

First we consider the linear Klein-Gordon equation unbounded in the x right-hand side

$$u_{tt}-u_{xx}=u+(t+1)x^{-2}[\cos\ln x+(1-x^2)\sin\ln x] \quad (10)$$

with the (oscillatory) initial conditions

$$u(x,0)=\sin\ln x, \quad (u)_t(x,0)=\sin\ln x. \quad (11)$$

[That is $\varphi(x,t)=(1+t)\sin\ln x$ and $h(x,t)=0$ in (3).]

We construct a homotopy which satisfies the relation

$$u_{tt}-(v_0)_{tt}+p\big[(v_0)_{tt}-u_{xx}-u \\ -(t+1)x^{-2}(\cos\ln x+(1-x^2)\sin\ln x)\big]. \quad (12)$$

Now substituting (8) into (12) and (11) and equating the coefficients of like powers of p, we get the system of equations

$$(u_0)_{tt}-(v_0)_{tt}=0, \quad u_0(x,0)=\sin\ln x, \\ (u_0)_t(x,0)=\sin\ln x, \quad (13)$$

$$(u_1)_{tt}+(v_0)_{tt}-(u_0)_{xx}-u_0 \\ =(t+1)x^{-2}[\cos\ln x+(1-x^2)\sin\ln x], \quad (14) \\ u_1(x,0)=0, \quad (u_1)_t(x,0)=0,$$

$$(u_2)_{tt}-(u_1)_{xx}-u_1=0, \\ u_2(x,0)=0, \quad (u_2)_t(x,0)=0, \quad (15)$$

etc. According to the alternative technique given by (7) for choosing the initial approximation v_0, we have $v_0=(t+1)\sin\ln x$. Thus solving (13)–(15) yields

$$u_0(x,t)=(t+1)\sin\ln x, \quad u_1=u_2=\ldots=0.$$

Hence, we have the exact solution $u(x,t)=(t+1)\sin\ln x$.

3.2. Example 2

Now we consider the equation

$$u_{tt}-u_{xx}+u^2=-xt^{-2}$$

with the initial conditions

$$\lim_{t\to 0+}(u(x,t)+x-x\ln t)=0, \\ \lim_{t\to 0+}(u_t(x,t)-xt^{-1})=0.$$

We construct a homotopy in the following form:

$$u_{tt}-(v_0)_{tt}+p\left[(v_0)_{tt}+\alpha u_{xx}+u^2+xt^{-2}\right].$$

By assuming the initial approximation $v_0=\varphi(x,t)+t\varphi_t(x,t)+L^{-1}(f(x,t)+h_{tt}(x,t))=x+x\ln t$, we have

$$(u_0)_{tt}-(v_0)_{tt}=0, \\ \lim_{t\to 0+}(u_0(x,t)+x-x\ln t)=0, \\ \lim_{t\to 0+}\left((u_0)_t(x,t)-xt^{-1}\right)=0,$$

$$(u_1)_{tt}+(v_0)_{tt}-(u_0)_{xx}+u_0^2=-xt^{-2}, \\ u_1(x,0)=0, \quad (u_1)_t(x,0)=0,$$

$$(u_2)_{tt}-(u_1)_{xx}+u_1^2=0, \\ u_2(x,0)=0, \quad (u_2)_t(x,0)=0,$$

t_i	Error
0.001	$-3.0131\cdot 10^{-5}$
0.002	$2.7642\cdot 10^{-5}$
0.004	$2.1041\cdot 10^{-4}$
0.006	$5.1204\cdot 10^{-4}$
0.008	$9.3319\cdot 10^{-4}$
0.009	$1.1887\cdot 10^{-3}$

Table 1. Error in $u_{tt}-u_{xx}+u^2+x/t^2=0$ at t_i.

etc. Solving these equations we obtain

$$u_0 = x + x\ln t,$$
$$u_1 = -3.75t^2x^2 + 2.5t^2x^2\ln t - 0.5t^2x^2\ln^2 t,$$
$$\begin{aligned}u_2 = &-0.286t^4 - 3.75t^2x^2 + 0.7789t^6x^4\\ &- 0.096t^6x^4\ln^3 t + 0.0083t^6x^4\ln^4 t\\ &+ (\ln t)(0.514t^4 - 0.925t^6x^4)\\ &+ (\ln^2 t)(0.435t^6x^4 - 0.083t^4).\end{aligned}$$

Some numerical values for the error of the approximate 3-term solution $u \simeq u_0+u_1+u_2$ are shown in Table 1.

3.3. Example 3

Finally, we consider the nonlinear sine-Gordon-type equation

$$u_{tt} - u_{xx} + \sin u = xt^{-1/2}$$

with the initial conditions

$$\lim_{t\to 0+}\left(u(x,0) - \frac{4}{3}x\sqrt{t^3}\right) = 0,$$
$$\lim_{t\to 0+}\left(u_t(x,t) - 2x\sqrt{t}\right) = 0.$$

We take $\sin u \simeq u - u^3/6 + u^5/120$ and construct a homotopy in the following form:

$$u_{tt} - (v_0)_{tt} + p\left[(v_0)_{tt} + u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{x}{\sqrt{t}}\right].$$

By assuming the initial approximation $v_0 = \frac{3}{4}x\sqrt{t^3}$, we have

$$(u_0)_{tt} - (v_0)_{tt} = 0,$$
$$\lim_{t\to 0+}\left(u_0(x,t) - \frac{4}{3}x\sqrt{t^3}\right) = 0,$$
$$\lim_{t\to 0+}\left((u_0)_t(x,t) - 2x\sqrt{t}\right) = 0,$$
$$(u_1)_{tt} + (v_0)_{tt} - (u_0)_{xx} + u_0 - \frac{u_0^3}{6} + \frac{u_0^5}{120} = \frac{x}{\sqrt{t}},$$
$$u_1(x,0) = 0, \quad (u_1)_t(x,0) = 0,$$
$$(u_2)_{tt} - (u_1)_{xx} + u_1 - \frac{u_1^3}{6} + \frac{u_1^5}{120} = 0,$$
$$u_2(x,0) = 0, \quad (u_2)_t(x,0) = 0,$$

etc. Solving these equations we obtain

$$u_0 = \frac{4}{3}x\sqrt{t^3},$$
$$u_1 = -\frac{16}{105}t^2x\sqrt{t^3} + \frac{128}{11583}t^2x^3\left(t^3\right)^{\frac{3}{2}} - \frac{512}{1177335}t^2x^5\left(t^3\right)^{\frac{5}{2}},$$
$$\begin{aligned}u_2 = {}&6.16\cdot 10^{-3}t^4x\sqrt{t^3} + \left(t^3\right)^{\frac{5}{2}}\\ &\cdot\Big(3.6\cdot 10^{-6}t^4x^5 - 7.2\cdot 10^{-5}t^4x^3\\ &+ 1.898\cdot 10^{-9}t^{12}x^5 - 5.132\cdot 10^{-10}t^{15}x^7\\ &+ 7.327\cdot 10^{-11}t^{18}x^9\Big) + \left(t^3\right)^{\frac{3}{2}}\\ &\cdot\Big(1.04\cdot 10^{-3}t^4x - 1.73\cdot 10^{-4}t^4x^3\\ &- 4.1\cdot 10^{-6}t^8x^3 + 5.7\cdot 10^{-7}t^{11}x^5\\ &- 4.43\cdot 10^{-8}t^{14}x^7 + 2.2\cdot 10^{-9}t^{17}x^9\\ &- 7.12\cdot 10^{-11}t^{20}x^{11} + 1.43\cdot 10^{-12}t^{23}x^{13}\\ &- 1.52\cdot 10^{-14}t^{26}x^{15}\Big).\end{aligned}$$

t_i	$u_{\text{approx.}}(0.01,t_i)$	Error
0.01	$4.2116\cdot 10^{-4}$	$4.7621\cdot 10^{-15}$
0.02	$3.7711\cdot 10^{-5}$	$2.7864\cdot 10^{-14}$
0.04	$1.0665\cdot 10^{-4}$	$1.2611\cdot 10^{-12}$
0.06	$1.9588\cdot 10^{-4}$	$1.1729\cdot 10^{-11}$
0.08	$3.0148\cdot 10^{-4}$	$5.7082\cdot 10^{-11}$
0.1	$4.2116\cdot 10^{-4}$	$1.948\cdot 10^{-10}$

Table 2. Error in $u_{tt}-u_{xx}+\sin u-\frac{x}{\sqrt{t}}=0$ at t_i.

Some numerical values for the error of the approximate 3-term solution $u \simeq u_0+u_1+u_2$ are shown in Table 2.

Note that the error in $u_{tt}-u_{xx}+\sin u - x/\sqrt{t}=0$ is small enough for the values of $x >> 0$ and/or $t >> 0$, for example, an error is less than $4.2373\cdot 10^{-9}$ for $x = 0.2$, $t = 0.1$, and an error is less than $3.1264\cdot 10^{-2}$ for $x = 0.8$, $t = 0.9$.

4. Conclusion

The HPM has been successfully employed to obtain the approximate analytical solutions of the generalized Klein-Gordon and sine-Gordon equations with unbounded right-hand side. The succes of the HPM depends on the proper choice of the initial approximation. We introduced a new, generalized type of the initial value problems to extend the scope of solvable problems.

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